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Toy models and stylized realities

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Abstract. I discuss the role of toy models as theoretical tools for understanding complex systems of interacting agents. I review some concrete examples, in order to illustrate how this approach is able to capture, though in an admittedly stylized way, the interactions and non-linearities which are responsible for the rich phenomenology observed in reality. This allows one to interpret the system's behavior in terms of phase transitions and critical phenomena.

PACS. 89.65.-s Social and economic systems – 05.70.Fh Phase transitions: general studies – 05.70.Jk Critical point phenomena – 89.65.Gh Economics; econophysics, financial markets, business and management

1 Introduction

Physics is an empirically grounded science, so it is no wonder that empirical studies have dominated the scene of physicists' studies in quantitative finance and economics. Indeed, the abundance of data recently made available in several disciplines has led to a new type of empirical science aiming at organizing such information into well defined statistical/empirical laws — or *stylized facts* (see e.g. [1,2]).

The implicit assumption is that such non-trivial laws are the consequence of interaction between the units of the system. However, inferring interactions from empirical data (e.g. from correlations) is a subtle issue. While physics can draw on the knowledge of microscopic laws to make such an inference, in socio-economic sciences or finance things are much more complex[1].

On their side, economists have realized the importance of going beyond *effective* non-interacting single agent theories such as the representative agent approach [4], as well as the need to properly deal with agents' heterogeneity and stochastic effects. The most interesting consequence of interaction in system with many interacting units are phase transitions[2]. These occur at points where single particle theories break down and separate regions with a qualitatively different collective behavior. The collective behavior beyond the transition cannot be reduced to that of simple non-interacting units and its description requires the introduction of new concepts and quantities (order parameters). Furthermore, in the strongly interacting phase, correlations may extend much further than interactions do[3], so that a naive inference of the latter from the former might be misleading.

In all these respects, theoretical modelling can give a significant contribution to the development of a fundamental theory as well as in guiding empirical research. Indeed theoretical models makes the relation between interaction and measurable collective behavior explicit (even though not univocal, in general). The experience which statistical mechanics has accumulated in understanding how thermal fluctuations and interaction determine different states of matter is very valuable in this respect. Indeed, the specific principle of physics — e.g. energy conservation and maximal entropy — which have clearly no validity beyond physics, play only a secondary role. As realized long ago [8], a similar statistical description of the states of interacting systems is possible even in other disciplines. Indeed, the relevance of phase transitions as a consequence of interaction has also been realized by some economists [6]. However, the point has not been taken further than some generalizations of known results on the

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[1] For example, as a corollary of market efficiency arguments, the presence of predictable patterns in financial time series (e.g. the well known January effect) signals the *absence* of traders exploiting such irregularities and their disappearance likely implies that traders have recognized such patterns and modified their behavior accordingly. In this case, empirical data on returns can at most say what agents are *not* doing. Section 2.2 discusses instead a case where the inference of agents' behavior is less problematic.

[2] Not all phenomena undergo phase transitions. An example in the recent literature on physics approach to economic phenomena are the models of wealth dynamics [5].

[3] The typical example is the emergence of long-range order in spin models, where even sites far apart which do not interact share the same orientation of their spins.

Ising models to model non-market interactions, and it has soon after been dismissed [7].

In this paper we shall briefly review a selection — admittedly strongly biased — of examples of phase transitions in socio-economic systems and models of financial markets. Such an endeavor in socio-economic sciences faces the additional hurdle of dealing with a much more complex reality than that of physics. The indifference of statistical laws to microscopic details justifies only partly the ruthless simplification which models need to make of reality. Indeed, rather than insisting on their realism, such toy models are best viewed as stylized realities where agents interact in well defined environments, capturing some key elements of the real systems. Only in some cases this leads to quantitative predictions for real systems. In all cases however, this approach provides a coherent picture which helps us organize the way in which we think about these systems and sheds light on what new phenomena we can expect. For this reason, in what follows the emphasis will be more on the overall picture and general considerations, than on technical details.

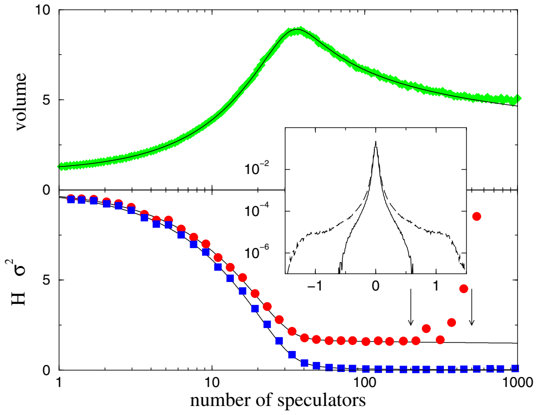

Fig. 1. Behavior of the Minority Game as a function of the reduced number of speculators $n_s = N_s/P$. Top: volume (number of active traders). Bottom: predictability H (squares) and global efficiency σ^2 (circles). The parameters of the GCMG are $\epsilon = 0.1$, $N_p = P$, $S = 1$ and $N_s P = 50\,000$. Averages were taken on 200 realizations. The inset shows the distribution of returns for 200 and 500 speculators (corresponding to the arrows) in numerical simulations ($P = 10$).

2 Critical phenomena in financial markets

Empirical analysis of financial markets data has uncovered non-trivial statistical features which are somewhat reminiscent of anomalous fluctuations in physics [1]. This raised the theoretical question of whether such stylized facts merely reflect the behavior of external factors (e.g. fundamentals, news arrival process) — as suggested by a naïve interpretation of market efficiency hypothesis — or whether they are the signature of a complex internal dynamics of a system of interacting traders.

To date, such stylized facts have been reproduced with a varying degree of success in many different ways – most having little to do with critical phenomena, rather arising as a consequence of a sort of *generic scale invariance* [20]. This suggests that stylized facts might not be sufficient to pin down a specific interaction mechanism. Still, some of these explanations have a larger theoretical content and provide a richer picture than others.

2.1 The Minority Game

High frequency fluctuations in returns are naturally related with speculative trading. This component of a financial market has sometimes been described as a *soup* of interacting trading strategies, subject to evolutionary selection, where the profitability of a strategy is not defined a priori, but rather depends on the composition of the soup [3].

The Minority Game (MG) [9] develops this intuition in a systematic way, pitting a number of adaptive agents against each other in the simplified market environment provided by the minority rule: Without entering into details (we refer to [10] for a brief account and to [9] for more details), agents are equipped with a small set of trading strategies which recommend them one of two actions (e.g. buy or sell) on the basis of some public information. The optimal choice is that placing the agent in the minority group and each agent selects one among his/her strategies according to their past performance. If speculators are also given the possibility of not trading if the game is not profitable (as in the so-called Grand-Canonical MG [9]), then the presence of *producers*, i.e. non-adaptive agents, becomes crucial in order to have some trading activity[4].

The typical behavior of such a market, in terms of trading volume V, information efficiency (i.e. predictability) H and volatility σ^2, as a function of the (reduced) number of speculators n_s is depicted in Figure 1. The behavior of H captures the main insight behind information efficiency arguments: A market with few speculators offers profitable gain opportunities ($H > 0$) to further speculators. However, as more speculators join the game, the market becomes more information efficient (i.e. $H \searrow n_s$). The number of speculators will stabilize when the market is nearly unpredictable ($H \approx 0$).

The inset shows that, precisely in the region where the market is nearly information efficient the distributions of returns acquires fat tails. Far from this region, i.e. when $H > 0$, returns are normally distributed. This implies that market (marginal) efficiency and anomalous fluctuations are two sides of the same coin, as they occur in the same region. A detailed study shows that indeed such fluctuations are due to finite size effects and disappear in the limit of markets with an infinite number of agents.

The MG also sheds light on the nature of the interactions between different types of traders in market ecologies and provides a number of surprising results. One is that the collective properties are largely independent of

[4] Interestingly, the presence of these two types of traders, a theoretical necessity in MG, has been confirmed empirically in the Spanish stock market [11].

whether the information which agents process is *endogenously* generated by the market process itself or whether it has an *exogenous* origin (depending e.g. on sun-spots activity) [12]. Another one is that, in efficient markets ($H = 0$) volatility (σ^2) decreases with the degree of randomness in agents' decision rule. Such an inverse relation between macroscopic fluctuations and microscopic noise is counter-intuitive in statistical physics. Equally striking is the fact that macroscopic fluctuations are independent of microscopic noise when $H > 0$ (see Sects. 3.3.5 and 3.4 of [9]).

But probably the most remarkable lesson of the MG is that concerning the role of the apparently innocent price-taking approximation on which agents' behavior — as well as most of financial engineering[5] — relies (see Chaps. 4 and 5 of [9]). In brief, market prices clearly depend on the actions of traders. However each single trader might regard himself as negligible with respect to "the market". This leads trader to take prices as fixed, neglecting their impact on them — the price-taking approximation. The MG reveals that this seemingly innocuous approximation leads agent to overestimate the worth of strategies they are not currently playing, and it accumulates over time causing agents to abandon optimal strategies for suboptimal ones. This strategy switching is what causes market volatility. Conversely, even approximately accounting for the market impact of their strategies, agents can collectively reduce significantly the market's volatility.

2.2 Multi-asset markets

Going from single asset to multi-asset markets, the relevant question becomes that of explaining the origin of financial correlations across stocks, which have been observed in empirical studies. The structure of correlations has been analyzed with several methods [16,17] and, in terms of its spectral decomposition, it is composed of three components: 1) noise background, which accounts for the bulk of the distribution, 2) economic correlations, which manifests in the few eigenvalues which leak out of the noise background and 3) the largest eigenvalue Λ — corresponding to the so-called *market mode* — which is well separated from the other ones as it accounts for a significant part of the correlations. The market mode describes the co-movement of stocks and, as shown in Figure 2a, exhibits a non-trivial dynamical behavior.

It is reasonable to assume that the properties of smaller eigenvalues are due to exogenous economic factors or to noise trading. The key question is then whether the wild fluctuations in Λ of Figure 2 are due to exogenous factors or to the internal non-linear dynamics of the market. This issue has been addressed in reference [18], starting from the following considerations: one important function of financial markets is that it allows traders to "dump" risk into the market by diversifying their investment across stocks, as postulated by portfolio theory [19]. This produces a flow of investment which is correlated across assets, with a sizeable component on the direction of the optimal portfolio. This, in turn, is likely to contribute to the correlation of returns, on which portfolio optimization depends. Such a feedback on financial correlations is captured by a symbolic equation for the covariance matrix $\hat{C}$:

$$\hat{C} = \hat{\Omega} + \hat{B} + \hat{F}\left(\hat{C}\right). \quad (1)$$

The idea is that asset correlations result from three different sources, corresponding to the three different components discussed above: 1) noise (including speculative) trading, which accounts for $\hat{\Omega}$; 2) fundamental trading, based on economics, represented by $\hat{B}$ and 3) the term $\hat{F}$ due to investment in risk minimization strategies, which itself depends on the financial correlations $\hat{C}$. Equation (1) depicts how "bare" economic correlations are dressed by the effect of financial trading and can be formalized in a simple phenomenological closed model for the joint evolution of returns and correlations [18]. This model predicts that, when the strength of the component of optimal portfolio investment increases, the market approaches a dynamic instability. It is precisely close to this point that the model reproduces time series of correlations with quite realistic properties (see Fig. 2b). Such a conclusion is confirmed by maximum likelihood estimation of the model's parameter on real market data.

The picture which this model offers is that of a market where risk minimization strategies contribute to the correlations they are trying to elude. The stronger the investment activity on these strategies, the more market's correlations grow, up to a point where the market enter a dynamically unstable phase. Interestingly, close to the

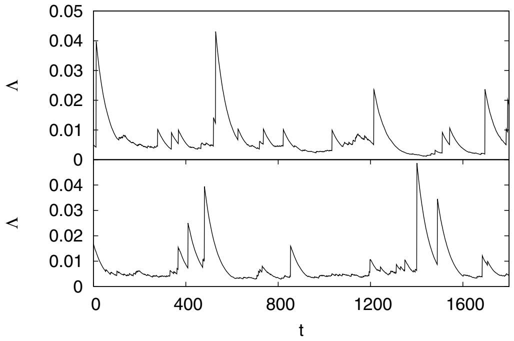

Fig. 2. Maximum eigenvalue of the exponentially averaged correlation matrix (over a typical timescale of approximately two months) as a function of time for the Toronto Stock exchange (top; data form Yahoo finance) and from numerical simulations of the phenomenological model of reference [18] (bottom).

[5] The relevance of market impact for large investors, as well as strategies for minimizing it by splitting large orders, has been studied e.g. in reference [13]. The resulting effects have also been claimed to be relevant to explain fat tails in the distribution of returns [14] and long term statistical dependencies in absolute returns [15]. We refer here to classical approaches, such as option pricing and portfolio theory where the investor is assumed small. Our point is that, even when the impact is negligible from the single agent point of view [13], the fact that many agents neglect it has sizeable consequences on market's dynamics.

phase transition the model develops anomalous fluctuations also in returns, which acquire a fat tailed distribution.

Within this simplified picture, real markets happen to operate close to the critical point corresponding to the dynamic instability. This is likely not a coincidence: indeed as far as a markets is far from the critical point, it offers stable correlations which allow traders to reliably use risk minimization strategies. However, as this opportunity is exploited more and more, the market approaches a point of infinite susceptibility where correlations become more and more unstable thus deterring further trading on these strategies.

2.3 Self-induced criticality?

Both the Minority Game and the phenomenological model of correlated assets relate the emergence of non-trivial fluctuations in financial markets to the critical phenomena taking place in the proximity of a phase transition. In both cases, a simple argument suggests that markets spontaneously evolve towards the critical state: as long as the density of traders is small enough, the market offers reliable opportunities (of speculation or risk minimization) which may attract further investors. Newcomers, however, make these opportunities less reliable and the market less attractive. Three points are worth noticing: (i) both cases show the fallacy of the price taking assumption on which most of financial engineering is based; (ii) the occurrence of an instability or critical point as the density of interactions increases is of the same nature of that discussed by May long ago [22] for complex systems in general and ecosystems in particular; and (iii) this scenario is reminiscent of Self-Organized Criticality (SOC) [21]. As in SOC, the presence of two different processes acting at well separated time-scales is essential. However, there are two important conceptual differences: first the concept of avalanches does not play a significant role and second the tendency to converge to the critical point is not due to external driving but to endogenous forces: The agents themselves, responding to incentives, drive the system close to the critical point. Under such endogenous forces, the density of interactions increases

In order to avoid confusion, it might be preferable to adopt a term different from Self-Organized Criticality in the present context.

We remark that a similar arguments, at the level of the economy for general equilibrium models, suggest that endogenous technological innovation should drive the economy close to a critical point [23]. The SOC paradigm has instead been advocated for model of bankruptcies [24,25].

3 Phase transitions in socio-economic systems

There are plenty of examples of statistical phenomena showing some sort of regularity, in the history of civilization. Of course modelling such phenomena is an highly speculative enterprise. For example, wars have come in all possible sizes during recent human history, causing from few hundreds to tens of millions causalities. Richardson [26] has observed in 1950 that the number of wars with more than N causalities are inversely proportional to the square root of N. Of course each war has its history of rights and wrongs. But such a regularity cannot be an accident and its similarity with critical branching processes or SOC is at least suggestive. Likewise, the cultural explosion which took place 40 000 years ago has been related to discontinuities which arise in simple model for the evolution of language in a population of interacting individuals [27]. The process by which a population converges to a set of agreed concepts has been also related to coarsening phenomena [28] and simple models of cultural diffusion exhibit phase transitions with a rich phenomenology [29].

3.1 The rise and fall of “liquid” societies

There is a growing consensus among social scientists on the relevance of the network dimension for social phenomena. Not only they are “embedded” in the underlying social network [30] but, reciprocally, the social network itself is largely shaped by the social processes taking place on it.

If we imagine links to carry profitable socio-economic interactions for the partners, one of the key function the network provides is that of preserving itself, helping agents to keep a dense network of interactions even in face of changing or volatile conditions. This intuition has been formalized recently in a class of stylized models [31,32] where a set of agents — be they individuals or organizations — establish bilateral interactions (links) when profitable. The favorable circumstances that led at some point to the formation of a particular link may later on deteriorate, causing that link’s removal. Hence volatility is a key disruptive element in the dynamics. On the other hand, the formation of new links may be constrained by the network architecture in many ways. For example, agents may rely on their social contacts to search further profitable opportunities and establish new links [31]. Alternatively, the profitability of a link might crucially depend on the similarity or proximity of the two partners (in terms of language, technological standards etc.) [32]. In both cases, the density of links in the network may display a non-linear behavior such as that displayed in Figure 3: as the networking effort — i.e. the rate at which agents try to form new links — increases from low values, the density of links initially does not increase substantially. Indeed, as long as the social network is sparse it does not provide either information on new possible partners nor does it achieves in establishing coordination or similarity across society. Beyond a critical threshold, however, a dense network of interactions with a giant component spanning a large fraction of the population emerges suddenly. Now the network enhances both search and coordination so that agents can efficiently replace obsolete links, keeping a high density of interactions. Such a network is resilient, because even if

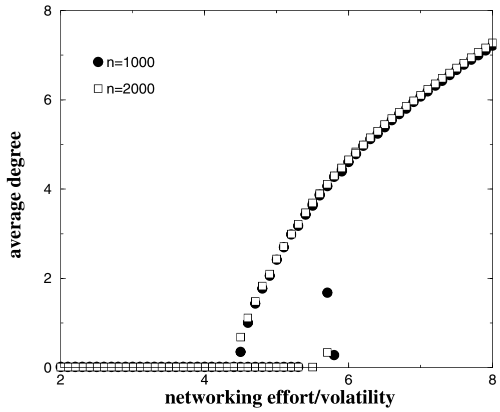

Fig. 3. Network density (average degree) as a function of the ratio between networking effort and volatility (ξ/λ) for the model of reference [31] ($\eta/\lambda = 0.01$).

the networking effort of agents decreases or if volatility increases, the society does not revert to a sparse network state but remains in a densely connected one.

This suggests that the dynamics of social network may feature sharp transition, resilience and phase coexistence whenever it provides functions which have a sizeable positive impact on the ability of agents to keep connected or to form new links. Such a picture is confirmed by anecdotic empirical evidence on the non-linear behavior of socio-economic networks in phenomena ranging from the spread of crime to the rise of industrial districts (see [32]).

4 Outlook

One would be tempted to say that socio-economic phenomena are currently changing at a rate which is much faster than that at which we understand things. For example, Nature has been there since ever, but it has taken centuries to develop a reasonable understanding of little parts of it. Many of the things which are traded nowadays in financial markets did not exist few decades ago, not to speak of internet communities. In addition, we face a situation in which the density and range of interactions are steadily increasing, thus making theoretical concepts based on effective non-interacting theories inadequate.

The emphasis which empirical analysis has acquired recently is a natural response to the need of organizing our description of complex phenomena. It is not clear whether computational (e.g. agent based) approaches, which have gained so much momentum recently, are contributing in the same direction. Computational approaches have been very useful in physics because the knowledge of microscopic laws constrains theoretical modelling in extremely controlled ways. This is almost never possible for socio-economic systems.

Phenomenological or toy models, which has been so useful in unveiling how interaction and thermal fluctuations compete in statistical physics, might be very valuable. As in the case of the Minority Game, these models open a wider window on the behavior of complex systems making it possible to locate "point-wise" empirical findings in a broader theoretical landscape. Even though stylized, such a reference, well controlled theoretical framework helps us shape our way of thinking about strongly interacting socio-economic systems and it may provide sources of inspiration for mining empirical data in search for fundamental theories and organizing principles.

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